Journal of Statistical Physics, Vol. 8, No. 2, 1973

On the Simple-Source Theory of Sound from Statistical Turbulence

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Received September 28, 1972

A theory for the generation of aerodynamic sound, stated in terms of convected simple sources and dipoles, is presented. The sources are found to depend upon convective derivatives of the hydrodynamic pressure within the turbulent source region. The results are similar to earlier efforts involving simple sources, sometimes called dilatational sources. The results are modified for effects involving measurements on moving flows. The theory shows explicitly the refractive effects of shear flow within the source region, as well as of temperature changes (if any) within the source region. The oscillating cylinder problem is discussed and the results of the present theory are found to agree with those obtained by Lauvstad using a matched asymptotic expansion for the same problem. The theory is used to predict the temperature dependence of sound power for hot jets.

KEY WORDS: Turbulence; random processes; noises.

1. INTRODUCTION

Lighthill[1-3] formulated the classical theory of aerodynamic sound by manipulation of the equation of continuity and the Navier–Stokes equations governing fluid flow. The equations are treated in such a way as to form a wave equation for the fluid density on the left side and to place all nonlinear, viscous and other terms on the right side. Thus, beginning with the equation of continuity,

$$(\partial \rho / \partial t) + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

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and the Navier–Stokes equations

$$(\partial/\partial t)\rho v_i + (\partial/\partial x_j)(\rho v_i v_j + p_{ij}) = 0 \tag{2}$$

with the fluid stress $p_{ij} = p\delta_{ij} - \eta_{ij}$, where η is the viscous stress, one finds

$$(\partial^2/\partial t^2)\rho - a_0{}^2\nabla^2\rho = (\partial^2/\partial x_i \partial x_j)T_{ij} \tag{3}$$

In Eq. (3) the source term T_{ij} is defined by

$$T_{ij} = \rho v_i v_j + p_{ij} - a_0{}^2\rho\delta_{ij} \tag{4}$$

The notation used here is common in the literature of aerodynamic sound (see, e.g., Refs. 1–3). In producing the wave equation (3), Lighthill subtracted the term $a_0{}^2\nabla^2\rho$ from both sides of the equation. This term involves the quantity a_0, which in applications is taken to be the ambient value of the speed of sound. However, it is interesting to note that (constant) a_0 is quite arbitrary in the formulation. The answers obtained would be correct for any choice of a_0.

Lighthill continues the development by solving the wave equation (3), following standard methods, and obtains the result for density changes within the medium written in terms of other fluid mechanical field quantities which are assumed to be known. Assuming that the fluid field quantities are known is technically equivalent to assuming knowledge of the full solution to the problem. Detailed information concerning the field quantities is usually not available. Following Lighthill, we are almost always driven to making estimates of the radiated sound. [It is relevant to observe that (3), of course, is not a full statement of the problem, being one scalar equation relating a number of unknown field variables.]

Curle[4] extended Lighthill's development to the situation where solid boundaries lie within the sound-generating, turbulent region. Starting from (3), one finds

$$\rho - \rho_0 = \frac{1}{4\pi a_0{}^2}\int_v \left[\frac{\partial^2}{\partial y_i\,\partial y_j}\,T_{ij}\right]\frac{d\mathbf{y}}{r} + \frac{1}{4\pi}\int_s \left[r^{-1}\frac{\partial\rho}{\partial n} + r^{-2}\frac{\partial r}{\partial n}\rho + (a_0 r)^{-1}\frac{\partial r}{\partial n}\frac{\partial\rho}{\partial t}\right] dS(\mathbf{y}) \tag{5}$$

where

$$r \equiv |\,\mathbf{x} - \mathbf{y}\,| \tag{6}$$

with $\mathbf{x}$ the field point. The square bracket [ ] indicates that one should use the retarded time: $t - r/a_0$. The first term in (5) gives mainly the "volume-sound," and the second gives the surface-sound. Integrating by parts and using (2), Curle, assuming S moves, at most, parallel to itself, found

$$\rho - \rho_0 = \frac{1}{4\pi a_0^2} \frac{\partial^2}{\partial x_i\, \partial x_j} \int_v [r^{-1}T_{ij}]\, d\mathbf{y} - \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int_s [r^{-1}P_i]\, dS(\mathbf{y}) \quad (7)$$

with P_i the components of the force per area exerted on the fluid by the solid boundaries.

In order to proceed, it is necessary to estimate the source terms. Lighthill does this and obtains for the intensity of the radiated sound the expression (which is typically valid for small-Mach-number flows and which is most often used for subsonic flows generally)

$$I_{\text{av}} = K\rho_0 V_0^8 a_0^{-5} D^2 x^{-2} \quad (8)$$

where D is the characteristic length (diameter) of the jet and $K \sim 10^{-5}$, with V_0 the jet speed. This is the average intensity at a great distance x. It may be said to arise from a combination of quadrupole sources; the source term in the volume integral in (5) is a quadrupole.

Curle proceeds similarly and obtains an estimate for the surface-sound field intensity. At frequencies low enough so that the wavelength is large compared with objects which are present Curle finds

$$I_s \sim \rho_0 V_0^6 a_0^{-3} D^2 x^{-2} \quad (9)$$

This is a dipole-generated sound. The estimate is for the second term in (7).

In Curle's expression the dipole nature of the sound source leads to an expression for the sound field intensity proportional to V_0^6 plus a volume source proportional to V_0^8.

These estimates are carried out in typical fluid mechanical ways based upon the characteristics appropriate to large-Reynolds-number flows.

It is the purpose of this paper to discuss an alternative formulation of aerodynamic sound theory based upon standard fluid mechanical approaches, which will express the radiated sound in terms of (convected) simple sources and dipoles within the turbulent volume of the fluid. This alternative discussion has been presented in part in earlier works.[5–7] It is known to be equivalent to Lighthill's treatment. The equivalence is discussed in Appendix C. It will be seen that the sources depend upon material derivatives of quantities within the flow and that any refracting characteristics, due to flow shear or due to flow temperature changes, are made explicit. In the classical treatment such refraction effects are, of course, implicit within the expressions.

2. THEORY OF AERODYNAMIC SOUND GENERATION IN TERMS OF SIMPLE SOURCES AND DIPOLES

We restrict the discussion to fluid flows where compressibility effects are slight, that is, where fractional density changes are small. The results presented here should apply to virtually any situation involving liquid flows, since in such a case fractional density changes are typically extremely small. We shall expand field quantities about what we shall call the "nearby incompressible flow" quantities. Suppose we are given the velocity field at a given time for a slightly compressible flow. We search for the incompressible flow field, satisfying any boundary conditions which may be imposed, which will minimize the square of the difference between the actual, slightly compressible flow field and the searched-for incompressible flow field. (This is reminiscent of—though here slightly more quantitative than—approximate treatments of aerodynamic problems for subsonic flow where one assumes that the fluid flow itself is incompressible. Such approximate treatments are of course common enough in airfoil lift and flutter theory.) After sufficient time the slightly compressible flow will wander from the neighboring incompressible flow. For homogeneous turbulence this (dimensionless) time is of order M^{-2}, where M is the Mach number; the characteristic time is of the order of the turbulence scale length divided by the velocity fluctuation.

We proceed by expanding all field quantities about their incompressible values. First adopt the notation wherein we write the exact compressible field quantities in (1) and (2) with an asterisk. Represent the nearby incompressible flow quantities with a subscript zero. Define the "passive," nonpropagating hydrodynamic density change ρ_1 :

$$\rho_1 \equiv a_0^{-2} p_0 \tag{10}$$

As suggested, p_0 is the incompressible pressure. The a_0 is conveniently taken to be the local value of the speed of sound. This local value may, of course, be a function of position if temperature changes are sufficiently great. We further suppose that the turbulence is statistically stationary. It is possible to define a time-averaged temperature within the turbulent region and we suppose that a_0 is the speed of sound corresponding to that average temperature. We shall neglect effects caused by fluctuations in the local value of the temperature. Complications, if any, arising from such temperature fluctuations could be treated in a development parallel to the present one, though it would, of course, be more complicated. Following the above-described notation we write the field quantities as follows:

$$\rho^* = \rho_0 + \rho_1 + \rho, \qquad \mathbf{v}^* = \mathbf{v}_0 + \mathbf{v}, \qquad p^* = p_0 + p \tag{11}$$

Now substitute the definitions (11) in (1) and (2), remembering that the starred quantities are the complete field functions in this notation. On substituting these expressions and recognizing that the incompressible flow quantities satisfy (1) and (2), we obtain for (1)

$$(\partial\rho/\partial t) + \rho_0\nabla\cdot\mathbf{v} + \mathbf{v}_0\cdot\nabla\rho = -(D/Dt_0)\,\rho_1 + \text{H.O.} \tag{12}$$

where $D/Dt_0 = \mathbf{v}_0\cdot\nabla$ is the substantial derivative following the nearby incompressible flow. Take (2) in the form with the density outside the derivatives and with the viscous force shown explicitly, and substitute (11) to find

$$\begin{aligned}\rho_0(\partial/\partial t)\,\mathbf{v} + \rho(\partial/\partial t)\,\mathbf{v}_0 + \rho_0[\mathbf{v}_0\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}_0] + \rho\mathbf{v}_0\cdot\nabla\mathbf{v}_0 \\ + \nabla p - \mu[\nabla^2\mathbf{v} + \tfrac{1}{3}\nabla\nabla\cdot\mathbf{v}] = -\rho_1(D/Dt_0)\mathbf{v}_0 + \text{H.O.}\end{aligned} \tag{13}$$

Here H.O. represents terms at least quadratic in $\mathbf{v}$, p, ρ, and ρ_1. In Appendix A we discuss the order of magnitude of the various field quantities involved. It is found there that

$$\rho_1/\rho_0 \sim v/v_0 \sim p/p_0 \sim M^2$$

within the turbulent flow and that $\rho/\rho_0 \sim M^4$ in the same region. Consequently, terms designated by H.O. are of higher order in the (assumed small) Mach number than are retained terms and are neglected. We have

$$(D/Dt_0)\,\rho + \rho_0\nabla\cdot\mathbf{v} = -(D/Dt_0)\rho_1 \tag{14}$$

and

$$\begin{aligned}\rho_0(D/Dt_0)\,\mathbf{v} + \rho(D/Dt_0)\,\mathbf{v}_0 + \rho_0\mathbf{v}\cdot\nabla\mathbf{v}_0 + \nabla p \\ - \mu[\nabla^2\mathbf{v} + \tfrac{1}{3}\nabla\nabla\cdot\mathbf{v}] = -\rho_1(D/Dt_0)\mathbf{v}_0\end{aligned} \tag{15}$$

Equations (14) and (15) are seen to be equations for the propagation of sound in moving media, where the motion is $\mathbf{v}_0$, with sound sources of two types: first, a simple source (a mass source) of (convected) strength per unit volume,

$$-(D/Dt_0)\,\rho_1 \equiv -(D/Dt_0)\,a_0^{-2}p_0 \tag{16}$$

and second, a dipole source per unit volume

$$-\rho_1(D/Dt_0)\,\mathbf{v}_0 = -a_0^{-2}p_0(D/Dt_0)\,\mathbf{v}_0 \tag{17}$$

The moving medium in general causes refraction effects controlled by the incompressible flow velocity $\mathbf{v}_0$ as seen in the left sides of (14) and (15). Suppose that in some problems the incompressible velocity is composed of a mean velocity (averaged in time) and a fluctuating velocity given by

$$\mathbf{v}_0 = \bar{\mathbf{v}}_0 + \mathbf{v}_0' \tag{18}$$

such that the mean velocity is considerably larger than the fluctuating (incompressible) velocity. Then we can write (14) and (15) in the approximate form

$$(\bar{D}/Dt_0)\,\rho + \rho_0\nabla\cdot\mathbf{v} = -(\bar{D}/Dt_0)\rho_1 \tag{19}$$

and

$$\rho_0(\bar{D}/Dt_0)\,\mathbf{v} + \rho(\bar{D}/Dt_0)\,\bar{\mathbf{v}}_0 + \rho_0\mathbf{v}\cdot\bar{\mathbf{v}}_0 + \nabla p - \mu[\nabla^2\mathbf{v} + \tfrac{1}{3}\nabla\nabla\cdot\mathbf{v}] = -\rho_1(D/Dt_0)\bar{\mathbf{v}}_0 \tag{20}$$

where

$$\bar{D}/Dt_0 \equiv (\partial/\partial t) + \bar{\mathbf{v}}_0\cdot\nabla \tag{21}$$

To complete the system of equations, we need the usual relationship between density and pressure change given in situations where the changes are approximately adiabatic:

$$p = a_0^2\rho \tag{22}$$

To simplify discussion, we can usually neglect the effect of viscosity on the sound field in (20). Further, we can treat the refractive effects of the moving medium (and variable a_0) as a separate problem and neglect $\mathbf{v}_0$ terms in the left sides of (19) and (20) (and let a_0 be constant).[3] In Appendix B we discuss the effect of hot (or cold) jets on the sound power produced.

Eliminate the term $\nabla\cdot\mathbf{v}$ between (19) and (20) and use (22) to find the wave equation with source

$$\frac{\partial^2}{\partial t^2}\rho - \nabla^2 a_0^2\rho = -\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} + \bar{\mathbf{v}}_0\cdot\nabla\right]a_0^{-2}p_0 + \nabla\cdot\left\{a_0^{-2}p_0\left[\frac{\partial}{\partial t} + \bar{\mathbf{v}}_0\cdot\nabla\right]\bar{\mathbf{v}}_0\right\} \tag{22'}$$

As it should, the simple source term, the first term on the right side of (22′), gives no sound for either a frozen, stationary flow (vanishing material derivative) or a flow which is stationary in the laboratory frame of reference (vanishing time derivative). The same is true for the dipole source term when the wave equation with source, (22′), is solved in the usual way (see Ref. 1).

The boundary conditions for the field variables at a rigid boundary or one moving parallel to itself are seen from our definitions to be

$$v_n = 0 \tag{23}$$

This is the appropriate condition for the inviscid sound problem.

[3] Strictly speaking, the second term in (21) should, for consistency, also be dropped; we retain it for its moving-source suggestive characteristic.

Consider a problem with rigid boundaries. To solve (22′) [or the less restricted (19) and (20)], we use the boundary condition (23) and the standard methods of acoustics (see Ref. 4).

An estimate of the sound field intensity from (22′) following standard methods gives

$$I \sim \rho_0 V_0{}^8 a_0^{-5} D^2 X^{-2} \tag{23'}$$

This is the same as Lighthill's result (8). Of course, there are differences of method of treatment as well between the two theories. We discuss the complete equivalence of results of the two theories in Appendix C for simplified aerosound problems.

The source terms considered here have a simple physical explanation. Suppose that we deal with an aerodynamic noise problem involving a jet source. Suppose that near the exit of the jet at one instant we have a region in which the hydrodynamic pressure is larger than the ambient value of the pressure. Enclose this positive pressure region with a control surface attached to the (incompressible) fluid motion. The pressure is higher in this region. The density is similarly higher by approximately ρ_1. This means that the control volume surrounded by the control surface has contracted with respect to its value just upstream.

A little farther downstream the pressure and density reduce within the control volume and the control surface expands. This constitutes a pulsation for the control surface whose normal component of velocity can be used as a boundary condition describing the radiation of sound form the control volume. The force as source acts similarly but produces dipole sound. The control surface encloses a region of fluid on which is exerted a force which is of the amount given by the right side of (20). This force acting on the control volume fluctuates as the hydrodynamic pressure and its attendant density change fluctuates while the control volume moves downstream. This fluctuating force constitutes a dipole sound source within the flow. These two sources, a simple source and a dipole source, generate sound which propagates through the shear region for the incompressible flow as given by (19) and (20). The sound beam refracted by that shear region follows the usual rules of sound propagation in moving media. If there were a temperature variation within the nearly incompressible flow, this would likewise refract the propagating sound. Its effect would appear when one substituted for either the pressure or the density using (22). Ribner and MacGregor[8] in their experiments dealing with refraction effects within turbulent jets have found that the source can be ascribed to a combination of simple source and simple dipole in amounts of about the same order, though these authors explain their results from a different viewpoint. From (22′) we see that the ratio of sound intensity generated by the convected simple source, the first term on the right side, to that from the convected simple dipole is $\bar{M}^{-2}$, where $\bar{M}$ is the *mean* Mach number, $\bar{M} = v_0/a_0$.[4]

3. DISCUSSION OF THE OSCILLATING CYLINDER PROBLEM

Consider now the problem of a circular cylinder oscillating sinusoidally about its fixed axis. This is a problem which can be solved in closed form for a cylinder in a viscous, incompressible fluid.[(9–11)] We apply the above simple source/dipole theory to this problem in the next section. The solution to the cylinder problem gives for the azimuthal velocity component (others vanish)

$$v_0 = V_0 \operatorname{Re} v_c(y, t) \tag{24}$$

with

$$v_c = e^{-i\omega t} H_1^{(1)}[(i\omega/\nu)^{1/2} y]/H_1^{(1)}[(i\omega/\nu)^{1/2} a] \tag{25}$$

and

$$p_0 = -\rho_0 \int_y^\infty y'^{-1} v_0^2(y', t)\, dy' \tag{26}$$

Re means take the real part. We have chosen the pressure reference at infinity; ω is the angular frequency of the oscillating cylinder and $H_1^{(1)}$ is the first-order Hankel function of the first kind. Also, a is the radius of the cylinder and V_0 is the amplitude of the speed of the cylinder surface.

The special character of this problem should be emphasized. It is a viscous flow so, e.g., the pressure for thin boundary layers is of order $(\nu/a^2\omega)^{1/2}\rho_0 V_0^2$, and thus is much smaller for large Reynolds numbers than pressures in more general problems.

4. APPLICATION OF THE SIMPLE SOURCE/DIPOLE THEORY OF AERODYNAMIC SOUND TO THE OSCILLATING CYLINDER PROBLEM

Consider the application of the aerodynamic sound theory represented by (14) and (15) to the oscillating cylinder problem. We suppose now that the fluid around the cylinder is slightly compressible with sound speed a_0 . For purposes of this theory-check we ignore fluid instabilities. We also neglect refraction effects caused by sound propagation through the incompressible

[4] To see this, first solve the wave equation with source, (22′), as in obtaining Lighthill's equation, first term of (5). Then integrate by parts, converting the divergence operation of the second source term in (22′) to $a_0^{-1}\, \partial/\partial t$. It is then evident that the ratio of the second source integral to the first is of order of magnitude $\bar{M}$. The ratio of sound intensities is the square of this quantity.

flow field. Likewise, as above, we neglect the effect of viscosity on the propagating sound. Furthermore, we adhere to the requirement that the Mach number of the flow field be small. The result is that (22′) becomes approximately

$$(\partial^2\rho'/\partial t^2) - a_0{}^2\nabla^2\rho' = -(\partial^2/\partial t^2)\, a_0^{-2}p_0 \tag{27}$$

with p_0 given for this oscillating cylinder problem by (26). In this approximation the source term is precisely the same as that proposed on somewhat less complete grounds in Ref. 9. For the purposes of this section we have replaced ρ of (22′) by ρ' as seen in (27). From the nature of p_0 it is seen that aside from possible transients (which we neglect by requiring that we examine the system only after it has been turned on for a considerable time), there is but one angular frequency in the sound field and it is 2ω. Then write

$$\rho' = \text{Re}\ \rho e^{-2i\omega t} \tag{28}$$

By simple manipulation of (27) using (24)–(26) and (28) and substituting for the source term of (27) we obtain

$$(\nabla^2 + 4k^2)\rho = 2\rho_0\omega^2 a_0^{-4} \int_y^\infty v_c{}^2(y', 0)\, y'^{-1}\, dy' \tag{29}$$

where $k = \omega/a_0$. The boundary condition is the familiar one for acoustics given in (23) and is to be applied at the surface of the cylinder. To simplify the discussion, we apply some restrictions as follows:

$$(\nu/\omega)^{1/2} \ll a \ll 2\pi/k \ll x \tag{30}$$

that is, we suppose that the viscous layer about the cylinder has a thickness which is much less than the radius of the cylinder, which in turn is much less than the sound wavelength. Furthermore, we suppose that the field point lies in the radiation region: x, the distance of the field point from the cylinder axis, is much greater than the sound wave length. From the nature of the incompressible solution given in (24) and (26) we see that the sound sources, the right side of (27), are confined to a region small compared with all the other lengths in the problem. In such a case the effect of the cylinder boundary upon the propagating sound is merely to double the source strength. We can use the free-space Green's function (remembering the source doubling)

$$(1/4i)\, H_0^{(1)}(2kr) \tag{31}$$

and ignore the presence of the cylinder. Thus, we solve the Helmholtz equation with source (29), use the Green's function (31), double the source strength to account for reflection at the rigid cylinder surface, substitute the resulting expression for ρ in (28), and use the asymptotic expansion for the Green's function to obtain for the sound field density change

$$\rho' = -\mathrm{Re}\left\{\frac{\rho_0}{4}\left(\frac{\pi a_0}{\omega x}\right)^{1/2}\left(\frac{v_0}{a_0}\right)^2 \frac{\nu\omega}{a_0^2}\, e^{i[2k(x-a_0t)+3\pi/4]}\right\} \tag{32}$$

Lauvstad[12] has used a matched asymptotic expansion method to calculate the sound generated when an oscillating cylinder is immersed in a slightly compressible fluid. His result, given in his equation (43) is, except for the sign and a factor of two, exactly the same as the result obtained here in (32). The method of the matched asymptotic expansion is widely believed to be the "correct," though often complicated, method for calculating such mixed regime problems as are typical of aerodynamic sound. It is concluded that the simple source theory outlined above has correctly calculated the generated sound for this oscillating cylinder problem.

It is possible to calculate the generated sound using the physical model of the generation process discussed above. That is, when the hydrodynamic pressure p_0 increases there is a corresponding increase in density near the surface of the cylinder. This causes, because of conservation of mass, a contraction of the edge of the boundary layer. There are opposite changes which occur when the hydrodynamic pressure decreases. This pulsation of the edge of the boundary layer can be used as a boundary condition for the normal component of the velocity for the generated sound field. If one pursues a treatment of this kind, he obtains again the result (32) for the sound field density change.

5. CONCLUSION

One way to state the purpose of subsonic aerosonic theory is the following. Consider a subsonic, turbulent jet (this is often extended to transonic); we wish to estimate the intensity of sound generated, through the use of essentially incompressible quantities. If we are driven to examine in detail the complicated compressible characteristics within the source region, the problem frequently becomes intractable. But to make such sound estimates, it seems desirable to place some emphasis upon the compressible characteristics of the fluid. This has been done here.

It has been shown that it is possible to formulate aerosonic theory in terms of a simple physical model. The simple source characteristic of the sound-generating region can be thought of as follows: Increases in the total hydrodynamic pressure within the jet cause what above have been termed passive changes in the density. These density increases lead to a contraction of a control surface around the positive pressure region because of conservation of mass. At a later instant all changes reverse somewhat downstream. Thus, if we think in terms of a control surface, it has been argued that the pulsation caused by hydrodynamic pressure changes of such a control surface become the cause of sound radiation. There is a similar dipole source term due to *forces* arising from density changes ρ_1. Both of these source mechanisms are convected in the jet. The simple source per volume is seen from (22′) to be approximately of order, except for a constant,

$$\rho_0 V_0^4 / a_0^2 D^2 \tag{33}$$

The theory is applied in some detail to the oscillating cylinder problem. Recent work by Lauvstad [11] has led to the calculation of the actual radiated sound for a slightly compressible fluid in the oscillating cylinder problem through the use of the method of matched asymptotic expansions. The theory described in the present paper, called the simple source/dipole theory, is used for the cylinder problem and it is shown that one obtains the sound intensity predicated by Lauvstad, except for a constant.

It would be interesting to apply these various methods to some other problem which can be calculated analytically. One which has some appeal is that of a pulsating sphere in an infinite fluid.

APPENDIX A. ESTIMATES OF FIELD QUANTITIES IN THE SOURCE REGION

It is our purpose here to estimate in typical situations the orders of magnitude of the field quantities ρ_1, ρ, and $\mathbf{v}$. First of all consider ρ_1 defined in (10). If we suppose, as is the case in typical applications, that the pressure p_0 is of order $\rho_0 v_0^2$ (within the turbulent source region), we have the familiar result

$$\rho_1 \sim \rho_0 a_0^{-2} v_0^2 \tag{A1}$$

that is, $\rho_1/\rho_0 \sim M^2$. Of course, it is true that in some special flows the pressure is not given by the estimate as suggested here. For example, the viscous flow about the oscillating cylinder discussed in this paper has a pressure which is also dependent on the viscosity.

Consider the orders of magnitude of the sound field quantities ρ and $\mathbf{v}$. First, ρ can be obtained for our purposes from (22′). Suppose we take as typical source term $-(\partial^2/\partial t^2)\rho_1$. Estimate the density change ρ resulting from this source term. Solving (22′) using this part of the source, we obtain

$$\rho(\mathbf{x}, t) = -(4\pi a_0^4)^{-1} (\partial^2/\partial t^2) \int r^{-1}[p_0]\, d\mathbf{y} \tag{A2}$$

so the dependence upon Mach number is given by $\rho/\rho_0 \sim M^4$. For the purpose of this estimate we restrict ourselves to problems involving free turbulence. Using the relationship given in (22) connecting ρ and p, sound field quantities, we have from (A1),

$$p/p_0 \sim M^2 \tag{A3}$$

as suggested in the theory section above. To estimate the sound field velocity $\mathbf{v}$, we can use the ordinary linearized acoustic equation obtained from (20), dropping the refractive terms involving $\mathbf{v}_0$. Comparing the lowest-order velocity term with the pressure term from that equation, we have

$$\rho_0\, \partial\mathbf{v}/\partial t \sim -\nabla p \sim -\nabla a_0^2 \rho \tag{A4}$$

Use for the time derivative the estimate v_0/L, where L is the scale of the turbulent process. Observe that the order of magnitude of the gradient has two possible values depending upon the nature of the problem and the region of the field which is being examined. The gradient can be order L^{-1} or of order $k/2\pi$:

$$v/v_0 \sim M^2 \quad \text{or} \quad M^3 \tag{A5}$$

using again the estimates just discussed for the characteristic frequencies for the turbulent process. Thus, v/v_0 is at least as small as M^2, as proposed in the text.

APPENDIX B. HOT JET AEROSOUND

One of the main applications of aerosonic theory is to the problem of sound production by jet engines. Evidently, the exhaust from such engines is typically at higher temperature than the surrounding air. Thus it is of interest to determine the dependence of sound production upon the temperature of the jet. As discussed, in the typical application we are presented with a convected simple source contained within the region of higher temperature (and consequently higher sound speed). Suppose that the sound speed and density in the source region are a_0 and ρ_0. We know, neglecting differences in gas composition (molecular weight) between the jet exhaust gases and the ambient, external atmosphere, that

$$a_0^2/a_1^2 = T_0/T_1 \tag{B1}$$

Here T_0 is the (absolute) temperature inside the jet and a_1 and T_1 are respectively the speed of sound outside and temperature outside the jet. For this discussion we neglect the effect of mean flow (moving sources) on the aerosound; in this connection, see Lighthill.[3] We further suppose that the fluctuation Mach number is small, so that the simple source of the radiated sound may be approximated by $-(\partial^2/\partial t^2)(p_0/a_0^2)$. Thus the resulting wave equation with source is given approximately by

$$(\partial^2 \rho/\partial t^2) - a_1^2 \nabla^2 \rho \cong -(\partial^2/\partial t^2)(p_0/a_0^2) \tag{B2}$$

where p_0 is approximately the hydrodynamic pressure inside the jet, sometimes referred to as the "pseudosound pressure" [see (22')]; other terms are of higher order. The simple source responsible for (B2) is given as the right side of (19); it is $-(D/Dt_0)\rho_1$. Let $\bar{\rho}_1$ be the ambient density outside the jet. For our low-frequency problem the amplitude of the radiated sound is increased by a factor $\bar{\rho}_1/\rho_0$ because of the increase in density going from the the hot jet to the exterior. We modify (22'), renaming the external ambient speed of sound a_1 and recognizing that the (passive) density change ρ_1 in the jet is given by $a_0^{-2}p_0$, where a_0 is the local value of the speed of sound within the source region (within the hot jet).

Proceed now to estimate the sound power radiated by the hot jet. We consider low frequencies (wavelength large compared with the jet size), so we may neglect refraction effects. The static pressure within the jet is of order

$$p_0 \sim \rho_0 v_0^2 \tag{B3}$$

where ρ_0 is a density within the hot jet and v_0 is the rms velocity fluctuation within the jet. The time derivative is of order v_0/D, where D is the diameter of the jet. Other estimates are made in the usual way (see Ref. 3) and one obtains for the total radiated sound power the expression for real, hot jets

$$P = K(\bar{\rho}_1 v_0^8 A/a_1^5)(a_1/a_0)^4 \tag{B4}$$

with $A = \pi D^2/4$; v_0 is the jet exit speed, where the constant K may be taken to be approximately 3×10^{-5}, supposing that the relative turbulence intensity of the hot jet is held constant or is independent of the temperature of the jet for the same flow speed. Note that the density effect in the pressure is cancelled by the density effect on the radiation efficiency described above. Using (B1), Eq. (B4) can be written (with $\bar{\rho}_1$ the ambient density)

$$\rho_0/\bar{\rho}_1 = T_1/T_0 \tag{B5}$$

and so (B4) becomes

$$P = K(\bar{\rho}_1 v_0^8 A/a_1^5)(T_1/T_0)^2$$

If the turbulence intensity level is held constant within the jets, two jets of different temperatures but the same flow speeds give a radiated sound power *inversely* proportional to the square of the absolute temperature of the jets. It is interesting that for jets for which these parameters are held constant, the hotter the jet, the less the sound produced.

APPENDIX C. COMPARISON OF THE SIMPLE-SOURCE THEORY WITH LIGHTHILL'S THEORY OF AERODYNAMIC SOUND

The equivalence of the simple source theory given in this report to Lighthill's presentation has been shown previously (see Refs. 5 and 6). It is nevertheless interesting to emphasize in a different way the relationship of the two theories.

Begin with Lighthill's equation for the source of aerodynamic sound,

$$(\partial^2/\partial t^2)\rho - a_0^2\nabla^2\rho = (\partial^2/\partial x_i\, \partial x_j)T_{ij} \tag{C1}$$

where

$$T_{ij} = (\rho_0 + \rho)\, v_i v_j + p_{ij} - a_0^2\rho\, \delta_{ij} \tag{C2}$$

and

$$p_{ij} = p\, \delta_{ij} - \eta_{ij} \tag{C3}$$

The last term is the viscous stress. Here ρ_0 is the average density, and p is the density change.

We shall consider here for simplicity an aerodynamic sound problem in which the jet is at approximately the same temperature as the ambient fluid. We neglect the effect of source motion. Furthermore, following Lighthill we may reasonably suppose that the viscous stresses are usually negligible in the sound production process. We have, accordingly, for the sound source tensor the expression

$$T_{ij} \cong \rho_0 v_i v_j + p\, \delta_{ij} - a_0^2\rho\, \delta_{ij} \tag{C4}$$

We are to suppose that the velocity fluctuations are small compared with the speed of sound a_0 within the jet. The adiabatic relation for pressure and density changes is

$$p = a_0^2\rho \tag{C5}$$

We show that there are two ways of pairing the terms of the source (C4) to be substituted in (C1). Lighthill's procedure for obtaining the source term is to use (C5) and obtain the approximate expression, right side of (C1),

$$(\partial^2/\partial x_i\, \partial x_j)\, \rho_0 v_i v_j \tag{C6}$$

Alternatively, for small Mach numbers we can use the approximate incompressibility of the fluid within the source (jet) region, taking the divergence of (2), and find that the quadratic velocity term cancels the pressure term when substituted in the right side of (C1). Paired in this fashion, we have for the wave equation with source the modified version of (C1) under restrictions discussed here

$$(\partial^2/\partial t^2)\,\rho - a_0{}^2\nabla^2\rho \cong -a_0{}^2\nabla^2\rho_1 \tag{C7}$$

recognizing that the density change in the source region is approximately $\rho_1 = a_0{}^2 p_0$ [see (10)]. Equation (C7) may be written under our restrictions as

$$(\partial^2/\partial t^2)\,\rho - a_0{}^2\nabla^2\rho \cong -\nabla^2 p_0 \tag{C8}$$

Solving (C8) following Lighthill, we find in the usual way that in the distant, radiation region the source term per volume is approximated by

$$-a_0^{-2}(\partial^2/\partial t^2)\,p_0 \tag{C9}$$

the result obtained in (22′) for low-Mach-number flows [or see (27)].

In low-Mach-number flows, it is seen that there are at least two ways of combining cancelling terms in the source in (C1) and (C2). If the pressure change is cancelled against $a_0{}^2$ times the density change, one obtains Lighthill's estimate of the source term involving the double divergence of the velocity fluctuations. On the other hand, if the approximately incompressible flow equation is used, the first two terms of the source T_{ij} cancel upon substitution in the right side of (C1) and one obtains the simple-source version of the theory discussed here. A final note is in order. From (C8) we see that the source may be obtained from the second space derivative of the pressure in the jet. The source *may* then be said to be a combination of (longitudinal) quadrupoles. The combination is, however, such as to give in the idealized case a spherically symmetric radiation pattern rather than the cloverleaf pattern expected from one isolated quadrupole source.

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